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Relationship between different equivalence relations in the space of standardizable systems

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Abstract

In the space $\mathscr{M}$ of generalized time-invariant linear systems $E\dot{x} = Ax + Bu$ we consider two equivalence relations, generalizing block-similarity of pairs $(A, B) \in M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C})$. Both equivalence relations can be defined by the action of Lie groups $\mathscr{G}_1 = Gl(n; \mathbb{C}) \times Gl(m; \mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ and $\mathscr{G}_2 = Gl(n; \mathbb{C}) \times \mathscr{G}_1$ acting on $\mathscr{M}$,

$$\begin{aligned} \alpha_1 : \mathscr{G}_1 \times \mathscr{M} &\to \mathscr{M} \\ ((P, R, U, V), (E, A, B)) &\to (P^{-1}EP + P^{-1}BU, P^{-1}AP + P^{-1}BV, P^{-1}BR) \end{aligned}$$

$$\begin{aligned} \alpha_2 : \mathscr{G}_2 \times \mathscr{M} &\to \mathscr{M} \\ ((P, Q, R, U, V), (E, A, B)) &\to (QEP + QBU, QAP + QBV, QBR). \end{aligned}$$

Restricting ourselves to $\mathscr{S}$, the open and dense set of standardizable systems, we prove that α_2 can be seen as the action of α_1 on the orbit space $\mathscr{S}/Gl(n; \mathbb{C})$. We also relate the corresponding miniversal deformations through the versal deformation of the orbit space.

Keywords: Equivalence relations; Lie group actions; Orbit spaces; Tangent spaces; Versal deformations

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1. Introduction

Let $\mathscr{M}$ be the differentiable manifold of triples of matrices (E, A, B) where $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, which represent generalized time-invariant linear systems in the form

$$E\dot{x}(t) = Ax(t) + Bu(t). \tag{1}$$

We will consider in $\mathscr{M}$ two different equivalence relations which generalize block-similarity of pairs of matrices: feedback-similarity and feedback-equivalence.

The triples (E, A, B) and (E', A', B') are said to be feedback-similar in the case where

$$(E', A', B') = (P^{-1}EP + P^{-1}BU, P^{-1}AP + P^{-1}BV, P^{-1}BR) \tag{2}$$

for some $P \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $U, V \in M_{m\times n}(\mathbb{C})$. That is to say,

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = P^{-1}\begin{pmatrix} E & A & B \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix},$$

that corresponds to apply in the system, the following standard transformations: basis change in the state space, basis change in the input space, feedback and derivative feedback.

The triples (E, A, B) and (E', A', B') are said to be feedback-equivalent in the case where

$$(E', A', B') = (QEP + QBU, QAP + QBV, QBR) \tag{3}$$

for some $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $U, V \in M_{m\times n}(\mathbb{C})$. That is to say,

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q\begin{pmatrix} E & A & B \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix}$$

that corresponds to apply in the system, the following standard transformations: basis change in the state space, basis change in the input space, feedback, derivative feedback and pre-multiplication by an invertible matrix.

In this paper we will relate both equivalence relations and restricting ourselves to the set $\mathscr{S}$ of triples defining standardizable systems (which is an open and dense set in $\mathscr{M}$) deduce an equivalence relation in the orbits space $\mathscr{S}/Gl(n; \mathbb{C})$ showing the relationship between the actions and miniversal deformations defined over $\mathscr{S}$ and over this orbits space.

2. Equivalence relations as induced by Lie group actions

Let us consider the following Lie groups: $\mathscr{G}_1 = Gl(n; \mathbb{C}) \times Gl(m; \mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$ and $\mathscr{G}_2 = Gl(n; \mathbb{C}) \times Gl(n; \mathbb{C}) \times Gl(m; \mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})$, acting on $\mathscr{M}$. The product $\bigstar_1$ in $\mathscr{G}_1$ is given by

$$\begin{aligned}&(P_1, R_1, U_1, V_1) \bigstar_1 (P_2, R_2, U_2, V_2)\\&\quad = (P_1P_2, R_1R_2, U_1P_2 + R_1U_2, V_1P_2 + R_1V_2)\end{aligned} \tag{4}$$

being $e_1 = (I_n, I_m, 0, 0)$ its unit element.

The product $\bigstar_2$ in $\mathscr{G}_2$ is given by

$$\begin{aligned}&(Q_1, P_1, R_1, U_1, V_1) \bigstar_2 (Q_2, P_2, R_2, U_2, V_2)\\&\quad = (Q_2Q_1, P_1P_2, R_1R_2, U_1P_2 + R_1U_2, V_1P_2 + R_1V_2)\end{aligned} \tag{5}$$

being $e_2 = (I_n, I_n, I_m, 0, 0)$ its unit element.

The actions of the Lie groups $\mathscr{G}_1$ and $\mathscr{G}_2$ on $\mathscr{M}$

$$\begin{aligned}\alpha_1 : \mathscr{G}_1 \times M &\to \mathscr{M}\\((P, R, U, V), (E, A, B)) &\to (P^{-1}EP + P^{-1}BU,\\&\qquad P^{-1}AP + P^{-1}BV, P^{-1}BR),\end{aligned} \tag{6}$$

$$\begin{aligned}\alpha_2 : \mathscr{G}_2 \times \mathscr{M} &\to \mathscr{M}\\((P, Q, R, U, V), (E, A, B)) &\to (QEP + QBU,\\&\qquad QAP + QBV, QBR)\end{aligned} \tag{7}$$

give rise to the equivalence relations in $\mathscr{M}$ which were called in §1 feedback-similarity and feedback-equivalence.

Notice that the $\mathscr{G}_2$-action can be seen as the composition of the $\mathscr{G}_1$-action and the $\mathscr{G}_0$-action,

$$\begin{aligned}\alpha_0 : \mathscr{G}_0 \times \mathscr{M} &\to \mathscr{M}\\(G, (E, A, B)) &\to (GE, GA, GB),\end{aligned} \tag{8}$$

where $\mathscr{G}_0 = Gl(n; \mathbb{C})$, whith the usual product and being the identity element $e_0 = I_n$.

That is,

$$\alpha_2((P, Q, R, U, V), (E, A, B)) = \alpha_0(QP, \alpha_1((P, R, U, V), (E, A, B))).$$

From now on, we will make use of the following notation: $g_0 = G \in \mathscr{G}_0$, $g_1 = (P, R, U, V) \in \mathscr{G}_1$, $g_2 = (P, Q, R, U, V) \in \mathscr{G}_2$, and $x = (E, A, B) \in \mathscr{M}$.

Given a triple $x_0 = (E_0, A_0, B_0) \in \mathscr{M}$ we define the maps

$$\alpha_{i_{x_0}}(g_i) = \alpha_i(g_i, x_0), \quad i = 0, 1, 2. \tag{9}$$

The equivalence class of the triple x_0 with respect to the $\mathscr{G}_i$-action, called the $\mathscr{G}_i$-orbit of x_0, is the range of the function $\alpha_{i_{x_0}}$ and is denoted by

$$\mathscr{O}_i(x_0) = \operatorname{Im} \alpha_{i_{x_0}} = \{\alpha_{i_{x_0}}(g_i) | g_i \in \mathscr{G}_i\}, \quad i = 0, 1, 2. \tag{10}$$

The stabilizer of x_0 under the $\mathscr{G}_i$-action is the null-space of the function $\alpha_{i_{x_0}} - x_0$. We denote it by

$$\mathrm{Stab}_i(x_0) = \{g_i \in \mathscr{G}_i | \alpha_{i_{x_0}}(g_i) = x_0\}, \quad i = 0, 1, 2. \tag{11}$$

Remark 2.1. The maps $\alpha_{i_{x_0}}$ are clearly differentiable and $\mathscr{O}_i(x_0)$, $\mathrm{Stab}_i(x_0)$ are smooth submanifolds of $\mathscr{M}$ and $\mathscr{G}_i$, respectively.

3. Description of tangent spaces to orbits

Let us denote by $T_{e_i}\mathscr{G}_i$ the tangent space to the manifold $\mathscr{G}_i$ at the unit element e_i for $i = 0, 1, 2$. It is known that

$$\begin{aligned}
&T_{e_0}\mathscr{G}_0 = M_{n\times n}(\mathbb{C})\\
&T_{e_1}\mathscr{G}_1 = M_{n\times n}(\mathbb{C}) \times M_{m\times m}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})\\
&T_{e_2}\mathscr{G}_2 = M_{n\times n}(\mathbb{C}) \times M_{n\times n}(\mathbb{C}) \times M_{m\times m}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})\\
&T_{x_0}\mathscr{M} = \mathscr{M}
\end{aligned}$$

Proposition 3.1. *Let $d\alpha_{i_{x_0}} : T_{e_i}\mathscr{G}_i \to \mathscr{M}$ be the differential of $\alpha_{i_{x_0}}$ at the unit element $e_i (i = 0, 1, 2)$. Then*

$$\begin{aligned}
&d\alpha_{0_{x_0}}(G) = (GE, GA, GB) \in \mathscr{M}, \quad G \in T_{e_0}\mathscr{G}_0.\\
&d\alpha_{1_{x_0}}((P, R, U, V)) = (EP - PE + BU, AP - PA + BV,\\
&\qquad\qquad BR - PB) \in \mathscr{M}, \quad (P, R, U, V) \in T_{e_1}\mathscr{G}_1.\\
&d\alpha_{2_{x_0}}((Q, P, R, U, V)) = (EP + QE + BU, AP + QA + BV,\\
&\qquad\qquad BR + QB) \in \mathscr{M}, \quad (Q, P, R, U, V) \in T_{e_2}\mathscr{G}_2.
\end{aligned} \tag{12}$$

Hermitian products in $\mathscr{M}$ and $T_{e_i}\mathscr{G}_i$ $(i = 0, 1, 2)$ we will deal with in this paper are the following ones:

$$\begin{aligned}
&\langle x_1, x_2\rangle_{\mathscr{M}} = \mathrm{tr}(E_1E_2^*) + \mathrm{tr}(A_1A_2^*) + \mathrm{tr}(B_1B_2^*), \quad x_i = (E_i, A_i, B_i) \in \mathscr{M},\\
&\langle y_1, y_2\rangle_0 = \mathrm{tr}(G_1G_2^*), \quad y_i = G_i \in T_{e_0}\mathscr{G}_0,\\
&\langle y_1, y_2\rangle_1 = \mathrm{trace}(P_1P_2^*) + \mathrm{tr}(R_1R_2^*) + \mathrm{tr}(U_1U_2^*) + \mathrm{tr}(V_1V_2^*),\\
&\quad y_i = (P_i, R_i, U_i, V_i) \in T_{e_1}\mathscr{G}_1,\\
&\langle y_1, y_2\rangle_2 = \mathrm{tr}(Q_1Q_2^*) + \mathrm{tr}(P_1P_2^*) + \mathrm{tr}(R_1R_2^*) + \mathrm{tr}(U_1U_2^*) + \mathrm{tr}(V_1V_2^*),\\
&\quad y_i = (Q_i, P_i, R_i, U_i, V_i) \in T_{e_2}\mathscr{G}_2,
\end{aligned} \tag{13}$$

where A^* denotes the conjugate transpose of the matrix A.

The adjoint linear mappings $d\alpha_{i^*_{x_0}} : \mathscr{M} \to T_{e_i}\mathscr{G}_i$ $(i = 0, 1, 2)$ are defined by the relation

$$\langle d\alpha_{i_{x_0}}(y), x\rangle_{\mathscr{M}} = \langle y, d\alpha_{i^*_{x_0}}(x)\rangle_i, \quad y \in T_{e_i}\mathscr{G}_i, \quad x \in \mathscr{M}. \tag{14}$$

Remark 3.1. It is well known that the maps $d\alpha_{i_{x_0}}$ and $d\alpha_{i^*_{x_0}}$ provide a simple description of the tangent spaces $T_{x_0}\mathscr{O}_i(x_0)$, $T_{e_i}\mathrm{Stab}_i(x_0)$ and their orthogonal complementary subspaces $(T_{x_0}\mathscr{O}_i(x_0))^\perp$, $(T_{e_i}\mathrm{Stab}_i(x_0))^\perp$:

$$T_{x_0}\mathscr{O}_i(x_0) = \text{Im } d\alpha_{i_{x_0}} \subset \mathscr{M},$$
$$(T_{x_0}\mathscr{O}_i(x_0))^{\perp} = \text{Ker } d\alpha_{i^*_{x_0}} \subset \mathscr{M},$$
$$T_i e_i \text{Stab}_i(x_0) = \text{Ker } d\alpha_{i_{x_0}} \subset T_{e_i}\mathscr{G}_i,$$
$$(T_{e_i}\text{Stab}_i(x_0))^{\perp} = \text{Im } d\alpha_{i^*_{x_0}} \subset T_{e_i}\mathscr{G}_i$$

(see [5,6], for example).

Given $x_0 \in \mathscr{M}$, the relationship among the tangent spaces $T_{x_0}\mathscr{O}_i(x_0)$ $(i = 0, 1, 2)$ is presented below.

Proposition 3.2. *The tangent spaces to the orbits of a triple of matrices $x_0 \in \mathscr{M}$ are related by*:

$$T_{x_0}\mathscr{O}_2(x_0) = T_{x_0}\mathscr{O}_0(x_0) + T_{x_0}\mathscr{O}_1(x_0)$$

Proof. If $v \in T_{x_0}\mathscr{O}_0(x_0) + T_{x_0}\mathscr{O}_1(x_0)$, $v = v_0 + v_1$ with $v_0 = (GE, GA, GB) \in T_{x_0}\mathscr{O}_0(x_0)$ and $v_1 = (EP - PE + BU, AP - PA + BV, BR - PB) \in T_{x_0}\mathscr{O}_1(x_0)$. Then $v = (EP + (G - P)E + BU, AP + (G - P)A + BV, BR + (G - P)B) \in T_{x_0}\mathscr{O}_2(x_0)$.

Conversely. Let v be a vector in $T_{x_0}\mathscr{O}_2(x_0)$, $v = (EP + QE + BU, AP + QA + BV, BR + QB)$. If $G = Q + P$, then $v = (GE, GA, GB) + (EP - PE + BU, AP - PA + BV, BR - BP) \in T_{x_0}\mathscr{O}_0(x_0) + T_{x_0}\mathscr{O}_1(x_0)$. $\square$

Corollary 3.1. *The orthogonal complementary subspaces to the tangent spaces to orbits of a triple of matrices x_0 are related by*:

$$T_{x_0}\mathscr{O}_2(x_0)^{\perp} = T_{x_0}\mathscr{O}_0(x_0)^{\perp} \cap T_{x_0}\mathscr{O}_1(x_0)^{\perp}.$$

From Proposition 3.1 it is not difficult to prove the following characterization.

Proposition 3.3. *Let $x_0 = (E, A, B) \in \mathscr{M}$ be a triple of matrices. Then*

$$(X, Y, Z) \in T_{x_0}\mathscr{O}_0(x_0)^{\perp} \Leftrightarrow EX^* + AY^* + BZ^* = 0,$$

$$(X, Y, Z) \in T_{x_0}\mathscr{O}_1(x_0)^{\perp} \Leftrightarrow \begin{cases} X^*E - EX^* + Y^*A - AY^* - BZ^* = 0, \\ X^*B = 0, \\ Y^*B = 0, \\ Z^*B = 0. \end{cases}$$

Taking into account Corollary 3.1 a description of $T_{x_0}\mathscr{O}_2(x_0)^{\perp}$ for $x_0 \in \mathscr{M}$ can be easily deduced.

Corollary 3.2. *Let $x_0 = (E, A, B) \in \mathscr{M}$ be a triple of matrices. Then*

$$(X, Y, Z) \in T_{x_0}\mathscr{O}_2(x_0)^{\perp} \Leftrightarrow \begin{cases} EX^* + AY^* + BZ^* = 0, \\ X^*E + Y^*A = 0, \\ X^*B = 0, \\ Y^*B = 0, \\ Z^*B = 0. \end{cases}$$

4. Miniversal deformations

In this Section will use the description of the orthogonal complementary subspaces to the tangent spaces to orbits obtained at the end of last Section for explicitly obtaining miniversal deformations.

First, we recall the definition of versal deformations. Let M be a smooth manifold.

Definition 4.1. Let $\mathscr{U}_0$ be a neighborhood of the origin of $\mathbb{C}^{\ell}$. A *deformation* $\varphi(\lambda)$ of x_0 is a smooth mapping

$$\varphi : \mathscr{U}_0 \to M$$

such that $\varphi(0) = x_0$. The vector $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathscr{U}_0$ is called the parameter vector.

The deformation $\varphi(\lambda)$ is also called *differentiable family* of elements of M.

Let $\mathscr{G}$ be a Lie group acting smoothly on M. We denote the action of $g \in \mathscr{G}$ on $x \in M$ by $g \circ x$.

Definition 4.2. The deformation $\varphi(\lambda)$ of x_0 is called *versal* if any deformation $\varphi'(\xi)$ of x_0, where $\xi = (\xi_1, \ldots, \xi_k) \in \mathscr{U}'_0 \subset \mathbb{C}^k$ is the parameter vector, can be represented in some neighborhood of the origin as

$$\varphi'(\xi) = g(\xi) \circ \varphi(\phi(\xi)), \quad \xi \in \mathscr{U}''_0 \subset \mathscr{U}'_0, \tag{15}$$

where $\phi : \mathscr{U}''_0 \to \mathbb{C}^{\ell}$ and $g : \mathscr{U}''_0 \to \mathscr{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0)$ is the identity element of $\mathscr{G}$. Expression (15) means that any deformation $\varphi'(\xi)$ of x_0 can be obtained from the versal deformation $\varphi(\lambda)$ of x_0 by an appropriate smooth change of parameters $\lambda = \phi(\xi)$ and an equivalence transformation $g(\xi)$ smoothly depending on parameters.

A versal deformation having minimal number of parameters is called *miniversal*.

The following result was proved by Arnold [1,2] in the case where $\mathrm{Gl}(n; \mathbb{C})$ acts on $M_{n\times n}(\mathbb{C})$, and was generalized by Tannenbaum [7] in the case where a Lie group acts on a complex manifold. It provides the relationship between a versal deformation of x_0 and the local structure of the orbit.

Theorem 4.1 (Tannenbaum).

1. *A deformation* $\varphi(\lambda)$ *of* x_0 *is versal if and only if it is transversal to the orbit* $\mathcal{O}(x_0)$ *at* x_0.
2. *Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of* x_0 *in* $\mathcal{M}$, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.

Let $\{v_1, \ldots, v_\ell\}$ be a basis of any arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ (for example, $(T_{x_0}\mathcal{O}(x_0))^\perp$).

Corollary 4.1. *The deformation*

$$x : \mathcal{U}_0 \subset \mathbb{C}^\ell \to \mathcal{M}, \quad x(\lambda) = x_0 + \sum_{i=1}^{\ell} \lambda_i v_i \tag{16}$$

is a miniversal deformation.

The Lie groups $\mathcal{G}_i (i = 0, 1, 2)$ act smoothly on $\mathcal{M}$. Thus we can apply the result above in order to deduce explicit miniversal deformations in the cases of the three actions considered in $\mathcal{M}$.

Proposition 4.1. *Let* $x_0 = (E, A, B)$ *be a triple of matrices. Let* $\{u_1, \ldots, u_r\}$, $\{v_1, \ldots, v_s\}$, $\{w_1, \ldots, w_t\}$ *be bases of the vector subspaces*

$$\begin{aligned}
F_0 &= \{(X, Y, Z) \in \mathcal{M} | EX^* + AY^* + BZ^* = 0\},\\
F_1 &= \{(X, Y, Z) \in \mathcal{M} | X^*E - EX^* + Y^*A - AY^* - BZ^* = 0, X^*B = 0,\\
&\qquad Y^*B = 0, Z^*B = 0\},\\
F_2 &= \{(X, Y, Z) \in \mathcal{M} | EX^* + AY^* + BZ^* = 0, X^*E + Y^*A = 0, X^*B = 0,\\
&\qquad Y^*B = 0, Z^*B = 0\}.
\end{aligned}$$

Then the maps defined by

$$\begin{aligned}
\varphi_0(\lambda_1, \ldots, \lambda_r) &= x_0 + \lambda_1 u_1 + \cdots + \lambda_r u_r,\\
\varphi_1(\lambda_1, \ldots, \lambda_s) &= x_0 + \lambda_1 v_1 + \cdots + \lambda_s v_s,\\
\varphi_2(\lambda_1, \ldots, \lambda_t) &= x_0 + \lambda_1 w_1 + \cdots + \lambda_t w_t
\end{aligned}$$

are miniversal deformations with respect to the $\mathcal{G}_0$*-action, the* $\mathcal{G}_1$*-action and the* $\mathcal{G}_2$*-action, respectively.*

5. Restriction to the set of standardizable systems

Let (E, A, B) be a generalized system as in Eq. (1)

$$E\dot{x}(t) = Ax(t) + Bu(t).$$

In the case where E is an invertible matrix, pre-multiplying this equation by E^{-1}, a standard system is obtained:

$$\dot{x}(t) = E^{-1}Ax(t) + E^{-1}Bu(t). \tag{17}$$

We will consider the subset $\mathscr{S} \subset \mathscr{M}$ of triples (E, A, B) representing generalized linear systems being the matrix E wether invertible or such that may become invertible by means of a derivative feedback. That is to say,

$$\mathscr{S} = \{(E, A, B) \in M \mid \exists F \in M_{m\times n}(\mathbb{C}) \text{ such that } E + BF \in Gl(n; \mathbb{C})\} \tag{18}$$

Remark 5.1. $(E, A, B) \in \mathscr{S}$ if and only if $\operatorname{rank}(E \quad B) = n$ (see, for exemple, [3,4]).

Proposition 5.1. *Let (E, A, B) be a triple in $\mathscr{S}$. Then $\mathscr{O}_0(E, A, B) \subset \mathscr{S}$, $\mathscr{O}_1(E, A, B) \subset \mathscr{S}$ and $\mathscr{O}_2(E, A, B) \subset \mathscr{S}$.*

Proof. If $(E, A, B) \in S$, there exists $F \in M_{m\times n}(\mathbb{C})$ such that $E + BF$ is invertible.

If $(E_1, A_1, B_1) \in \mathscr{O}_0(E, A, B)$, $E_1 = GE$ and $B_1 = GB$ for some $G \in \mathscr{G}_0$. Then $E_1 + B_1F$ is invertible.

If $(E_1, A_1, B_1) \in \mathscr{O}_1(E, A, B)$, $E_1 = P^{-1}EP + P^{-1}BU$, $B_1 = P^{-1}BR$ for some $(P, R, U, V) \in \mathscr{G}_1$, then $E_1 + B_1(R^{-1}FP - R^{-1}U)$ is straightforward invertible.

Finally, if $(E_1, A_1, B_1) \in \mathscr{O}_2(E, A, B)$, $E_1 = QEP + QBU$, $B_1 = QBR$, for some $(Q, P, R, U, V) \in \mathscr{G}_2$, then $E_1 + B_1(R^{-1}FP - R^{-1}U)$ is also invertible. $\square$

Lemma 5.1. *Let (E, A, B) be a triple in $\mathscr{M}$. Then $\operatorname{rank}(E \quad B)$ is invariant under the $\mathscr{G}_i$-actions $(i = 0, 1, 2)$.*

Proof. The statement follows from

$$\begin{aligned}\operatorname{rank}(E \quad B) &= \operatorname{rank} G(E \quad B)\\ &= \operatorname{rank} P^{-1}(E \quad B)\begin{pmatrix} P & 0\\ U & R\end{pmatrix}\\ &= \operatorname{rank} Q(E \quad B)\begin{pmatrix} P & 0\\ U & R\end{pmatrix}. \qquad \square\end{aligned}$$

Proposition 5.2. *$\mathscr{S}$ is an open and dense set of $\mathscr{M}$ and $\mathscr{G}_0$ acts freely over $\mathscr{S}$.*

Proof. Taking into account that $(E, A, B) \in \mathscr{S}$ if and only if $\operatorname{rank}(E \quad B) = n$ and

$$G(E \quad A \quad B) = (E \quad A \quad B),$$

if and only if $G = I_n$, the statement follows. $\square$

Proposition 5.3. *The orbits space $\mathscr{S}/\mathscr{G}_0 = \{\mathscr{O}_0(E, A, B) | (E, A, B) \in \mathscr{S}\}$ has a differentiable structure such that the natural projection $\pi : \mathscr{S} \to \mathscr{S}/\mathscr{G}_0$ is a submersion. In fact $\pi : \mathscr{S} \to \mathscr{S}/\mathscr{G}_0$ is a principal bundle with structure group $\mathscr{G}_0$.*

Proof. It is sufficient to prove (see, for example, [8], Theorem (2.9.10)) that

(i) $\Gamma = \{((E, A, B), (E', A', B')) | E' = GE, A' = GA, B' = GB$ for some $G \in \mathscr{G}_0\}$ is closed in $\mathscr{S} \times \mathscr{S}$, and
(ii) the map $\gamma : \mathscr{S} \times G \to \Gamma$ defined by

$$\gamma((E, A, B), G) = (E, A, B), ((GE, GA, GB))$$

is a homeomorphism.

To prove (i), we consider the set

$$\begin{aligned}\Delta = \{((E, A, B), (E', A', B')) | E' = GE,\\ A' = GA, B' = GB \text{ for some } G \in \mathscr{G}_0\}\end{aligned}$$

which is closed in $M^*_{n\times 2n+m} \times M^*_{n\times 2n+m}$, where $M^*_{n\times 2n+m}$ denotes the subset of complex matrices having n rows and $2n + m$ columns with maximal rank. Notice that $S \subset M^*_{n\times 2n+m}$.

Clearly $\Gamma = \Delta \cap (\mathscr{S} \times \mathscr{S})$ and Δ is closed in $M^*_{n\times 2n+m} \times M^*_{n\times 2n+m}$.

Concerning (ii), it is clear that γ is onto and the injectivity follows from Proposition 5.2. □

Remark 5.2. Let (GE, GA, GB) be any triple in $\mathscr{O}_0(E, A, B)$. Then

$$\begin{aligned}&P^{-1}(GE \quad GA \quad GB)\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix}\\ &\quad = (P^{-1}GP)P^{-1}(E \quad A \quad B)\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix} \in \mathscr{O}_0(E_1, A_1, B_1),\end{aligned}$$

where

$$(E_1, A_1, B_1) = (P^{-1}EP + P^{-1}BU, P^{-1}AP + P^{-1}BV, P^{-1}BR).$$

Then an action of $\mathscr{G}_1$ on $\mathscr{S}/\mathscr{G}_0$ may be defined

$$\begin{aligned} \alpha : \mathscr{G}_1 \times S/\mathscr{G}_0 &\to S/\mathscr{G}_0 \\ ((P, R, U, V), \mathscr{O}_0(E, A, B)) &\to \mathscr{O}_0(P^{-1}EP + P^{-1}BU, \\ &\qquad P^{-1}AP + P^{-1}BV, P^{-1}BR) \end{aligned} \tag{19}$$

That is to say, $\alpha((P, R, U, V), \pi(E, A, B)) = \pi(\alpha_1(P, R, U, V), (E, A, B))$. Moreover, $\mathscr{O}(\mathscr{O}_0(E, A, B)) = \cup_{(E_i, A_i, B_i) \in \mathscr{O}_1(E, A, B)} \mathscr{O}_0(E_i, A_i, B_i)$.

Proposition 5.4. *For any $(E, A, B) \in \mathscr{S}$, the orbit $\mathscr{O}_2(E, A, B)$ is diffeomorphic to $\mathscr{O}(\mathscr{O}_0(E, A, B))$, the orbit of $\mathscr{O}_0(E, A, B)$ under the α-action.*

Finally, we will consider $\mathscr{G}_2$-action on $\mathscr{S}$ and the action α on $\mathscr{S}/\mathscr{G}_0$. We will prove that a versal deformation of any element $\mathscr{O}_0(E, A, B) \in \mathscr{S}/\mathscr{G}_0$ can be obtained through a versal deformation of $(E, A, B) \in S$.

Theorem 5.1. *Let $\psi : \mathscr{U} \to S$ be a deformation of $(E, A, B) \in \mathscr{S}$. Then*

(a) *$\pi \circ \psi$ is a deformation of $\pi(E, A, B) = \mathscr{O}_0(E, A, B) \in \mathscr{S}/\mathscr{G}_0$.*
(b) *Any deformation φ of $\mathscr{O}_0(E, A, B)$ can be written as $\pi \circ \psi$, for some deformation ψ of (E, A, B) in $\mathscr{S}$.*
(c) *ψ is versal if and only if $\varphi = \pi \circ \psi$ is versal.*

Proof

(a) $\pi \circ \psi : \mathscr{U} \to \mathscr{S}/\mathscr{G}_0$ is differentiable and $(\pi \circ \psi)(0) = \mathscr{O}_0(E, A, B)$.
(b) Since $\pi : \mathscr{S} \to \mathscr{S}/\mathscr{G}_0$ is a principal bundle, there exists a local section σ of π.
(c) Let us assume that φ is versal at $\mathscr{O}_0(E, A, B)$. Then any other deformation $\varphi'(\xi)$ of $\mathscr{O}_0(E, A, B)$, can be represented in some neighborhood of the origin in the following form:

$$\varphi'(\xi) = \alpha((P(\xi), R(\xi), U(\xi), V(\xi)), \varphi(\phi(\xi))) = \alpha(g(\xi), \varphi(\phi(\xi))) \tag{20}$$

for some $g : \mathscr{U}_0'' \to \mathscr{G}_1$, $\mathscr{U}_0'' \subset \mathscr{U}_0'$.

Taking into account (ii), there exist ψ and ψ' such that $\varphi = \pi \circ \psi$ and $\varphi' = \pi \circ \psi'$.

$$(\pi \circ \psi')(\xi) = \alpha(g(\xi), (\pi \circ \psi)(\phi(\xi))) = \pi(\alpha_1(g(\xi), \psi(\phi(\xi))) \ \forall \xi \in \mathscr{U}_0''.$$

Since $\psi'(\xi)$ and $\alpha_1(g(\xi), \psi(\phi(\xi)))$ are in the same orbit with respect to α_0. Then for all ξ there exists an unique $G(\xi)$ such that

$$\alpha_0(G(\xi), \alpha_1(g(\xi), \psi(\phi(\xi))) = \psi'(\xi).$$

The map $\gamma : \mathscr{U}_0''' \to \mathscr{G}_0$, $\mathscr{U}_0''' \subset \mathscr{U}_0''$, defined by $\gamma(\xi) = G(\xi)$ is smooth and

$$\begin{aligned}\psi'(\xi) &= \alpha_2((P(\xi), G(\xi)P(\xi)^{-1}, R(\xi), U(\xi), V(\xi)),\\ &\qquad (E(\phi(\xi)), A(\phi(\xi)), B(\phi(\xi))))\\ &= \alpha_2((P(\xi), G(\xi)P(\xi)^{-1}, R(\xi), U(\xi), V(\xi)), \psi(\phi(\xi))).\end{aligned}$$

Then ψ is versal.

Conversely. Let us assume that ψ is versal.

Let $\varphi' : \mathscr{V} \to \mathscr{S}/\mathscr{G}_0$ be any deformation of $\pi(E, A, B)$ and $\sigma : \mathscr{S}/\mathscr{G}_0 \to \mathscr{S}$ be a local section. Then $\psi' = \sigma \circ \varphi'$ is a deformation of (E, A, B) and there exists

$$\begin{aligned} \gamma : \mathscr{V}' &\to \mathscr{G}_2 \\ \xi &\to (P(\xi), Q(\xi), R(\xi), U(\xi), V(\xi)) \end{aligned}$$

such that

$$\begin{aligned} \psi'(\phi(\xi)) = \alpha_2(\gamma(\xi), \psi(\phi(\xi))) &= \alpha_2((P(\xi), Q(\xi), \\ &\quad R(\xi), U(\xi), V(\xi)), \psi(\phi(\xi))) \\ &= \alpha_0(Q(\xi)P(\xi), \alpha_1((P(\xi), R(\xi), U(\xi), V(\xi)), \psi(\phi(\xi)))). \end{aligned}$$

Thus

$$\begin{aligned} \varphi'(\xi) &= (\pi \circ \psi')(\xi) \\ &= \pi(\alpha_1((P(\xi), R(\xi), U(\xi), V(\xi)), \psi(\phi(\xi)))) \\ &= \alpha((P(\xi), R(\xi), U(\xi), V(\xi)), (\pi \circ \psi)(\phi(\xi))) \\ &= \alpha((P(\xi), R(\xi), U(\xi), V(\xi)), \varphi(\phi(\xi))) \quad \forall \xi \in \mathscr{V}'. \end{aligned}$$

Then φ is versal. □

Corollary 5.1. *With the same notation as in the Theorem, φ is miniversal if and only if ψ is miniversal.*

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